Training course for the MAT paper in November

Topic 0: sample notes on the MAT course

Outline. These sample notes contain 2 short Problems 1.1, 3.5 (with general hints and solutions) and long Problems 1.13 (with general and step-by-step hints) and 6.13 (without hints). Please try to solve all the problems independently. If necessary, use the general hints and step-by-step hints. We shall e-mail a solution to first homework Problem 1.13 with our feedback on your script. All topics are ordered in the increasing order of difficulty according to our past experience. If Problem 1.13 is too easy for you, we are happy to mark Problem 6.13 for free. If you find mistakes or typos, e-mail Dr Olga Anosova, master.maths.tutor@gmail.com.

The *prime factorisation* theorem says that any positive integer has a factorisation $n = {p_1}^{a_1} p_2^{a_2} \dots p_k^{a_k}$ into finitely many prime factors $p_1, p_2, \dots, p_k$. Here $a_1, a_2, \dots, a_k \geq 1$ are integers. A prime decomposition of an integer n is unique up to a re-arrangement (a permutation) of the prime factors.

Problem 1.1. Find integers a, b such that $\dfrac{8^a}{12^{a-b}} \cdot \dfrac{9^{a+3b}}{6^{a+b}}$ is integer.

Problem 1.13. Let us consider 2012 towns in the UK. On the 1st day the weather was luckily dry in all 2012 towns. After a strange cyclone on the 2nd day, every second town became wet. So on the 2nd day there were 1006 dry towns $(1, 3, 5, \dots, 2011)$ and 1006 wet towns $(2, 4, 6, \dots, 2012)$.

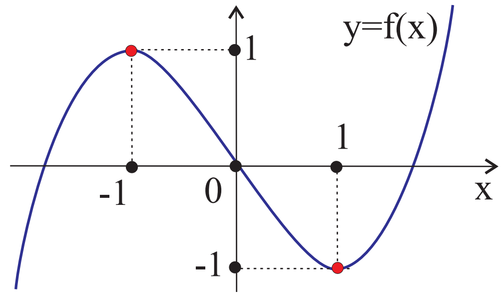
(i) Another cyclone on the 3rd day changed the weather (from dry to wet or vice versa) in every third town. For instance, town 3 became wet, town 6 became dry and so on. How many towns are dry now?

(ii) One more strange cyclone on the 4th day changed the weather in every fourth town. How many towns are dry now?

(iii) The strange cyclones continued their behaviour. The n-th cyclone on the n-th day changed the weather in every n-th town. After the 2012-th cyclone is the 2012-th town dry or wet? Justify your every conclusion.

Problem 3.5. Find the greatest value of $f(x) = (4\cos^4(5x-6)-3)^2$.

Problem 6.13. (i) The graph $y = f(x)$ of some function is shown on the right. Sketch the graphs of the functions $y = f(-x)$, $y = f(1-x)$, $y = 1-2f(1-x)$ and label their local extrema.



(ii) Sketch the graphs of $y = e^{-x^2}$, $y = e^{4x-x^2}$. Label any local extrema.

Please try to solve each problem above first independently for at least 5 min (10 min for a long problem) before using the general hints below.

General hints on Problem 1.1 (other methods are possible)
Simplify and rewrite the given expression in terms of only prime factors.

General hints on Problem 1.13 (other methods are possible)
(i) You may start from a small number of towns, e.g. 6 towns.

(ii) The 4th cyclone affects blocks of 12 towns in a similar way.

(iii) Generalise your observations and patterns from (i) and (ii).

General hints on Problem 3.5 (other methods are possible)
Find the upper and lower bounds of $4\cos^4(5x-6)-3$ over all real x.

All training problems are modified from past MAT papers (Maths Admissions Test) and occasionally from relevant maths questions in PAT papers (Physics Admissions Test). MAT 2001/2002 denote two specimen papers. Regular MAT papers started in 2006. In original MAT papers, the first ten short questions (collected in Problem 1) are numbered from A to J. Please pay your extra attention to the final (bullet •) tips after each solution.

Solution to Problem 1.1 (other methods are possible)
The prime factorisation theorem allows you to represent the given product in a *standard form* by rewriting all numbers as products of primes:

$$\frac{8^a}{12^{a-b}} \cdot \frac{9^{a+3b}}{6^{a+b}} = \frac{2^{3a} \cdot 3^{2(a+3b)}}{2^{2(a-b)}3^{a-b} \cdot 2^{a+b}3^{a+b}} = \frac{2^{3a}3^{2a+6b}}{2^{3a-b}3^{2a}} = 2^b 3^{6b}.$$

To guarantee that $2^b 3^{6b}$ is integer, the exponent b should be non-negative, while a can be any integer, because the result doesn't depend on a.

• The method works for any powers and prime factors instead of $2, 3$.

Solution to Problem 3.5 (other methods are possible)
The given function is too complicated to differentiate it. You may remember the bounds ± 1 of simplest trigonometric functions. For any real x, the function $\cos^4(5x-6) \geq 0$ varies between 0 and 1. Then $4\cos^4(5x-6)-3$ varies between -3 and $4 \cdot 1 - 3 = 1$. Hence the square of this difference is between 1 and $(-3)^2 = 9$. The greatest value 9 is attained when we have $4\cos(5x-6) = 0$, which has infinitely many real solutions.

• There are many easy elegant methods to find a maximum or minimum value of a function. Differentiating is a rather brute-force approach.

Step-by-step hints on Problem 1.13 (other methods are possible)

Step [1]. We start from 6 towns and encode the dry weather by 1.

Step [2]. If a town becomes wet, we may encode this event by 0. Then apply the 2nd cyclone to the code of 6 towns and produce a new code.

Step [3]. How does the 3rd cyclone affect the code of 6 towns?

Step [4]. Extend the weather pattern from Step [3] to 12 towns, which is repeated on 3rd day since 12 is the lowest common multiple of $2, 3, 4$.

Step [5]. Without writing 2012 codes, use the periodic pattern in blocks of 12 towns to count the number of dry towns after the 3rd cyclone.

Step [6]. Similarly to Step [5], what towns are dry on the 4th day?

Step [7]. What town (and how) should we monitor to answer (iii)?

Step [8]. What cyclones affect the weather in the last 2012-th town?

Step [9]. Factorise 2012 in primes. Why is the largest factor prime?

Step [10]. Count all divisors $k > 1$ (not only prime) of the number 2012.