Training course for the STEP I exam in mathematics

Topic 0: factors of numbers and polynomials

Outline. The sample notes contain (1) training Problem 1.1 with general hints, step-by-step hints and a detailed solution; (2) homework Problem 1.3 with general and step-by-step hints; (3) our feedback on a fictional attempt at Problem 1.1 on the last page. Please try to solve training Problem 1.1 first independently. If you are not sure how to start, look at the general hints. If you are still stuck, look at the step-by-step hints. If you need more help, e-mail us what steps you completed and we shall give extra hints.

Similarly approach homework Problem 1.3, first independently, then use hints if needed. We are happy to e-mail a step-by-step solution to Problem 1.3 with our feedback on your script for free. If you find mistakes or typos, e-mail Dr Olga Anosova, master.maths.tutor@gmail.com.

The regular notes on each topic of the STEP I course contain two training problems with general hints, step-by-step hints, solutions and one homework problem with general hints and step-by-step hints. Detailed solutions always follow our step-by-step hints in the notes, also review common mistakes of students and provide final tips in similar cases.

An integer $k > 0$ is called *prime* if the only positive factors of k are 1 and k. To check whether a given integer k is prime, consider all primes only up to $\sqrt{k}$ (fast enough without a calculator). For divisibility by 3, use the criterion: a number k is divisible by 3 if and only if the sum of its digits is divisible by 3. Indeed, let $k = \overline{d_1 d_2 \dots d_n}$, where $d_1, d_2, \dots, d_n$ are successive digits of k. Then $k = 10^{n-1}d_1 + \dots + 10d_{n-1} + d_n$, e.g. $\overline{d_1 d_2} = 10d_1 + d_2$ or $12 = 10 \cdot 1 + 2$. Subtract the sum of the digits: $k - (d_1 + \dots + d_{n-1} + d_n) = 9 \dots 9d_1 + \dots + 9d_{n-1}$. This difference is divisible by 3. So k and its sum of digits are divisible by 3 only simultaneously. The similar criterion works for 9. Since $1 + 1 + 1 = 3$, then 111 is divisible by 3, but not by 9. Since $3 + 4 + 5 + 6 = 2 \cdot 9$, then 3456 is divisible by 9.

Problem 1.1. **(i)** Find an integer solution (a, b) of the equation $a^2 = b^2 + 1551$ such that $a, b > 0$ and b is small.

(ii) Factorise 1551. Find all integers $a, b > 0$ such that $a^2 = b^2 + 1551$.

The *prime factorisation* theorem says that any positive integer has a factorisation $k = 2^{a_2}3^{a_3} \dots p^{a_p}$ into finitely many prime factors $2, 3, \dots, p$. Here $a_2, a_3, \dots, a_p$ are non-negative integer powers. A prime decomposition of k is unique up to a rearrangement of factors. A simple proof: if k is prime, then k is the only prime factor of k. Otherwise $k = a \cdot b$ for some integers a, b that are between 2 and $\frac{k}{2}$. Then check if a, b are primes and decompose them similarly. This process stops in finitely many steps, because at each step new numbers a, b decrease at least by the factor of 2.

Problem 1.3. A factor q of an integer k is called *proper* if $1 < q < k$.

(i) Prove that $36 = 2^2 \cdot 3^2$ has exactly 7 proper factors. Find how many other integers of the form $2^n \cdot 3^m$ have exactly 7 proper factors.

(ii) Let k be the smallest positive integer that has exactly 321 proper factors. Find a prime factorisation of k.

General hints on Problem 1.1 (other solutions are possible).

(i) Try small integers $b > 0$ to get a complete square $b^2 + 1551$.

(ii) Any integer solution (a, b) gives a factorisation of 1551.

General hints on Problem 1.3 (other solutions are possible).

(i) Find all proper factors of the numbers 36, 2^n, 3^m and $2^n \cdot 3^m$.

(ii) Let $k = 2^{a_2}3^{a_3} \dots p^{a_p}$. Similarly to (i), what can you say about the powers $a_2, a_3, \dots, a_p$ if the number of *all* positive factors of k is 323?

Step-by-step hints on Problem 1.1 (other solutions are possible).

Step [1]. Check integers $b = 1, 2, 3, 4, 5, 6, 7$. Find the value of a.

Step [2]. Recognise a difference of squares in $a^2 = b^2 + 1551$.

Step [3]. Use the solution (a, b) from Step [1] to factorise 1551.

Step [4]. Write down a full prime factorisation of the number 1551.

Step [5]. How many decompositions into two factors does 1551 have?

Step [6]. Each decomposition of 1551 into two factors gives (a, b).

Step-by-step hints on Problem 1.3 (other solutions are possible).

Step [1]. List all positive factors of 6. Highlight all proper factors.

Step [2]. Justify why 2^n has $n - 1$ proper factors for any $n \geq 1$.

Step [3]. As in Step [2], how many proper factors does $3^m \geq 3$ have?

Step [4]. List all seven proper factors of $2^2 \cdot 3^2$ as required in (i).

Step [5]. Any factor of $2^n \cdot 3^m$ has the form $2^p \cdot 3^q$ for $p \leq n$, $q \leq m$.

Step [6]. To count factors of $2^n \cdot 3^m$, count pairs (p, q), $0 \leq p \leq n$, $0 \leq q \leq m$. Justify that $2^n \cdot 3^m$ has $(n + 1)(m + 1) - 2$ proper factors.

Step [7]. Find all integers of the form $2^n \cdot 3^m$ with 7 proper factors.

Step [8]. Write down a prime factorisation of an integer k in terms of powers $a_2, a_3, a_5, \ldots,$ where a_2 is the power of the prime factor 2 etc.

Step [9]. How many positive factors does $k = 2^{a_2} 3^{a_3} \ldots p^{a_p}$ have?

Step [10]. If $k = 2^{a_2} 3^{a_3} \ldots p^{a_p}$ has 321 proper factors, write down an equation for $a_2, a_3, \ldots, a_p$. What values can $a_2, a_3, \ldots, a_p$ take?

Step [11]. If the number k is smallest, keep only two powers a_2, a_3.

Step [12]. For each decomposition of 323 into integer factors, write down the powers a_2, a_3 and the number k. Choose the smallest number k.

Solution to Problem 1.1 (modified question 1 STEP I 2006).

Step [1]. After trying small integers b, we find that $b = 7$ gives a complete square: $7^2 + 1551 = 49 + 1551 = 1600 = 40^2$, so $a = 40$.

Step [2]. You may recognise the difference of squares in the given equation:

$$a^2 = b^2 + 1551, \quad a^2 - b^2 = 1551, \quad (a - b)(a + b) = 1551.$$

Step [3]. The solution $a = 40$, $b = 7$ from Step [1] gives the following factorisation in Step [2]: $1551 = (40 - 7)(40 + 7) = 33 \cdot 47$.

Step [4]. The factor 47 is prime (check only potential factors $2, 3, 5$). Since $33 = 3 \cdot 11$, then the prime factorisation is $1551 = 3 \cdot 11 \cdot 47$.

Step [5]. Apart from the decomposition $1551 = 33 \cdot 47$ into two factors from Step [3], find three more: $1551 = 11 \cdot 141 = 3 \cdot 517 = 1 \cdot 1551$.

Step [6]. Each of the three decompositions from Step [5] gives a solution. For instance, if $a - b = 11$, $a + b = 141$, then $a = \dfrac{11 + 141}{2} = 76$ and $b = 76 - 11 = 65$. Similarly, if $a - b = 3$, $a + b = 517$, then $a = 260$, $b = 257$. Finally, if $a - b = 1$, $a + b = 1551$, then $a = 776$, $b = 775$. So there are four positive integer solutions of the equation $a^2 = b^2 + 1551$, namely $(a, b) = (40, 7), (76, 65), (260, 257), (776, 775)$.

Common mistakes that were made by many students in Solution 1.1

- A simple solution $a = 40$, $b = 7$ was hard to find, because the square $1600 = 40^2$ wasn't recognised as a close approximation to 1551.
- Some integer solutions were missed, because not all decompositions of 1551 into two positive integer factors were taken into account.

Useful tips from Solution 1.1 (they can be used in other problems)

- Start solving a complicated equation by trying simple values.
- Recognise the difference of squares: $a^2 - b^2 = (a - b)(a + b)$.
- A non-prime number k has many factorisations including $1 \cdot k$.
- The method of Solution 1.1 works for any integer instead of 1551.

This is our sample feedback on fictional homework by John Smith (Problem 1.1), where we comment on many common mistakes, logical gaps and writing style

Dear John, thank you for your homework: 12/20 marks

1.1(i) $a, b > 0$ integers $a^2 = b^2 + 1551$

a good style is to explain how you start, e.g. "we'll find a small integer b>0"

$b = 1 \rightarrow a^2 = 1^2 + 1551 = 1552$, $a = \sqrt{1552}$ not integer

the arrow -> isn't a logical implication, but used for convergence or for a function

$b = 2 \rightarrow a^2 = 2^2 + 1551 = 1555$, $a = \sqrt{1555}$ no

a faster way is to look for a square a^2 close enough to 1551

...

(between 1500 and 1600), aha here is a square: 1600=40^2

moreover, you can't use calculator at a real STEP exam (or at Oxbridge interview)

$b = 7 \rightarrow a^2 = 7^2 + 1551 = 1600$, $a = \sqrt{1600} = 40$ yes!

the logical implication => below is wrong, because the equation doesn't imply that

$a^2 = b^2 + 1551 \Rightarrow a = 40, b = 7$ integer solutions

the only integer solution is (a,b)=(40,7), safer: "(a,b)=(40,7) is an integer solution"

7/8 marks

1.1(ii) Other $a, b > 0$? $a^2 = b^2 + 1551$

Yes, a STEP problem usually contains clear hints how to start, read carefully

The hint in the problem is to factorise 1551

Without a calculator: 1551 is divisible by 3, because so is the sum 12 of digits

Trying primes on a calculator: $1551 = 3 \cdot 517$

The logical implication => below is correct, but can be confusing, we should always

$a^2 = b^2 + 1551 \Rightarrow a^2 - b^2 = 3 \cdot 517$ stuck???

try writing equivalent equations to avoid false roots: a comma is ok instead of <=>

after looking at the step-by-step hints in the notes

You may always look at our hints even after completing your solution independently

$(a^2 - b^2) = (a-b) \cdot (a+b) = 3 \cdot 517 \Rightarrow a - b = 3$, $a + b = 517$

The logical implication above leads to only one solution (a,b), it's wrong: 2/6 marks

We should consider all possible decompositions of 1551 into 2 factors (a-b)*(a+b)

$b = a - 3$, $a + a - 3 = 517$, $2a = 520$, $a = 260$, $b = 257$

For instance, (a,b)=(40,7) in (i) comes from 1551=33*47, two more decompositions are 11*141 and 1*1551 (often forgotten)

Answer: $a = 260$, $b = 257$ and $a = 40$, $b = 7$ from (i)

Two more integer solutions are (a,b)=(76,65) and (776,775) 3/6 marks