Topic 0: sample notes on the STEP II and III courses

The sample notes contain (1) first homework Problem 1.4 on the STEP II course with general hints and step-by-step hints, (2) homework Problem 5.4 on the STEP II course with general hints, (3) homework Problem 8.4 on the STEP III course without any hints. If you find any mistakes or typos, please e-mail Dr Olga Anosova, master.maths.tutor@gmail.com.

Please try to solve homework Problem 1.4 first independently. If you are not sure how to start, look at the general hints. If you are still stuck, look at the step-by-step hints. If you need more help, please e-mail and we'll provide extra hints. We are happy to e-mail a step-by-step solution to homework Problem 1.4 with our feedback on your script for free.

The regular notes on each topic of the STEP II and III courses contain **three** training problems with general hints, step-by-step hints, solutions and **one** homework problem with general hints and step-by-step hints.

Detailed solutions always follow the step-by-step hints, review common mistakes and provide final tips in similar cases. See a solution with our feedback on a fictional script in the sample notes on the STEP I course: http://www.master-maths.co.uk/courses/STEP1-sample-notes.pdf

The *period* of a sequence $a_1, a_2, a_3, \dots$ is the minimum integer $k > 0$ such that $a_{n+k} = a_n$ for all $n \geq 1$. If such k doesn't exists, then the sequence is called *aperiodic* (has no period). Any sequence with period 1 is constant.

Problem 1.4. The sequence $r_1, r_2, r_3, \dots$ is defined by $r_1 = 3$ and the recurrence relation $r_{k+1} = d - \dfrac{18}{r_k}$ for $k \geq 1$, where d is fixed.

(i) Find the values of the real parameter d such that the sequence r_k is

(a) constant,

(b) periodic with period 2,

(c) periodic with period 4.

(ii) In the case $d = 12$ show that $r_k \geq 6$ for all $k \geq 2$. Assuming that in this case the given sequence r_k converges to a limit L, find the value of L.

To find all rational solutions $x > y > 0$ of $x^2 - y^2 = 1$, factorise: $(x-y)(x+y) = 1$. Let $r = x + y > 0$ be rational, then $x - y = \frac{1}{r}$. Hence $x = \frac{1}{2}\left(r + \frac{1}{r}\right) = \frac{r^2+1}{2r}$ and $y = \frac{r^2-1}{2r}$ for any rational $r > 0$.

Problem 5.4. (i) Let $x^3 + y^3 = n$ and $x + y = \frac{n}{k}$ for integers $n, k > 0$ and rational $x > y > 0$. Show that $k^3 - n^2 = 3kx(kx - n)$ and $4n^2 > 4k^3 \geq n^2$. Find all rational roots $x > y > 0$ of the cubic equation $x^3 + y^3 = 20$.

(ii) Find all rational $x > y > 0$ such that $x^3 + y^3 = 19$, $x + y$ is integer.

Problem 8.4. (i) For continuous functions $f(t)$ and $g(t) \neq 0$, find

$$\int \frac{f(t)g'(t) - f'(t)g(t)}{\big(g(t)\big)^2}\, dt.$$

(ii) By choosing suitable functions $f(t)$ and $g(t)$, find $\int \frac{5 + 4t - 3t^2}{\left(t^2 - 2t + 3\right)^2}\, dt$. Can you choose another function $f(t)$ and get a different answer?

(iii) Find the general solution of the differential equation:

$$(\sin\alpha - 2\cos\alpha + 3)y'(\alpha) = (\cos\alpha + 2\sin\alpha)y(\alpha) + 3(\cos\alpha - \sin\alpha - 1).$$

General hints on Problem 1.4 (other methods are possible).

(i)(a) It suffices to consider the first two elements r_1 and r_2. Why?

(i)(b) Consider the first three elements. Exclude the previous case.

(i)(c) Consider the first five elements. Exclude the two previous cases.

(ii) Induction. Assume that the sequence becomes constant to find L.

General hints on Problem 5.4 (other methods are possible).

(i) Factorise $n = x^3 + y^3$ to prove that $n^2 \leq 4k^3 < 4n^2$. If $n = 20$, there are only 3 possible integers k leading to only one rational pair $x > y > 0$.

(ii) $x + y = \frac{n}{k}$ from (i) is irrelevant, the new condition: $x + y = l$ is integer. Similarly to (i), estimate $l^3 = (x + y)^3$ in terms of $x^3 + y^3 = n = 19$.

Step-by-step hints on Problem 1.4 (other methods are possible).

Step [1]. Compute the first three elements r_1, r_2, r_3 of the sequence.

Step [2]. The sequence is constant if the first two elements are equal.

Step [3]. Find the parameter d when the elements $r_1 = r_2$ are equal.

Step [4]. The sequence has period 2 if and only if $r_3 = r_1 \neq r_2$.

Step [5]. Find two values of the parameter d when $r_1 = r_3$.

Step [6]. Choose the single correct answer in part (b) of (i).

Step [7]. Write down and simplify the elements r_4, r_5 in terms of d.

Step [8]. The sequence has period 4 only if $r_1 = r_5$, $r_2 \neq r_1 \neq r_3$.

Step [9]. Simplify the equation $r_1 = r_5$ to get a cubic polynomial.

Step [10]. Using a root from Step [3], factorise the cubic polynomial.

Step [11]. For $d = 12$, prove that $r_k \geq 6$ by induction on $k \geq 2$. Start from $k = 2$. The inductive step: use $r_k \geq 6$ to get $r_{k+1} = 12 - \frac{18}{r_k} \geq 6$.

Step [12]. To find the limit, assume that $r_{k+1} = r_k$ for all large k.